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A generalized Lichnerowicz formula, the Wodzicki residue and gravity

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Abstract

We prove a generalized version of the well-known Lichnerowicz formula for the square of the most general Dirac operator $\widetilde{D}$ on an even-dimensional spin manifold associated to a metric connection $\widetilde{\nabla}$. We use this formula to compute the subleading term $\Phi_1(x, x, \widetilde{D}^2)$ of the heat-kernel expansion of $\widetilde{D}^2$. The trace of this term plays a key role in the definition of a (euclidian) gravity action in the context of non-commutative geometry. We show that this gravity action can be interpreted as defining a modified euclidian Einstein–Cartan theory.

Keywords: Non-commutative geometry; Lichnerowicz formula; Dirac operator; Heat-kernel expansion; Wodzicki residue; Gravity

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1. Introduction

When attempting to quantize the electron in 1928, Dirac introduced a first-order operator the square of which is the so-called wave operator (d'Alembertian operator). Later on, in the hands of mathematicians generalizations of this operator, called 'Dirac operators' – evolved into an important tool of modern mathematics, occurring for example in index theory, gauge theory, geometric quantization, etc.

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More recently, Dirac operators have assumed a significant place in Connes' non-commutative geometry [C] as the main ingredient in the definition of a K-cycle. Here they encode the geometric structure of the underlying non-commutative 'quantum-spaces'. Thus disguised, Dirac operators re-enter modern physics, since non-commutative geometry can be used, e.g. to derive the action of the Standard Model of elementary particles, as shown in [CL,K1]. Initially, it remains unclear whether it was possible also to derive the Einstein–Hilbert action of gravity using this approach. Further, it was a Dirac operator which proved to be the key to answer this question. According to Connes [C], the 'usual' Dirac operator D on the spinor bundle S associated to the Levi-Civita connection on a four-dimensional spin manifold M is linked to the euclidian Einstein–Hilbert gravity action via the Wodzicki residue of the inverse of D^2. This was shown in detail in [K2]. A further question that naturally arises is the dependence of this result from the chosen Dirac operator D. In other words, does $\mathrm{Res}(\widetilde{D}^{-2})$ change if $\widetilde{D}$ is a Dirac operator on S different from the 'usual' one ?

In this paper we answer this question affirmatively. Moreover, in Section 3, we compute the lagrangian of an appropriately defined gravity action

$$I_{\mathrm{GR}}(\widetilde{D}) := -\frac{2}{2^n(2n-2)}\,\mathrm{Res}(\widetilde{D}^{-2n+2}) \tag{1.1}$$

for the most general Dirac operator $\widetilde{D}$ associated to a metric connection $\widetilde{\nabla}$ on a compact spin manifold M with $\dim M = 2n \geq 4$. We proceed as follows: According to the main theorem of [KW], there is a relation between the Wodzicki residue $\mathrm{Res}(\hat{\Delta}^{-n+1})$ of a generalized laplacian $\hat{\Delta}$ on a hermitian bundle E over M and $\Phi_1(x, x, \hat{\Delta})$, which denotes the subleading term of its the heat-kernel expansion. It is well known that, given any generalized laplacian $\hat{\Delta}$ on E, there exists a connection $\hat{\nabla}^E$ on E and a section F of the endomorphism bundle $\mathrm{End}(E)$, such that $\hat{\Delta}$ decomposes as

$$\hat{\Delta} = \Delta^{\hat{\nabla}^E} + F. \tag{1.2}$$

However, this has a slight flaw. The decomposition (1.2) neither provides any method to construct the connection $\hat{\nabla}^E$ nor the endomorphism F explicitly. Nevertheless, it is exactly this endomorphism $F \in \Gamma(\mathrm{End}\,E)$ which fully determines the subleading term $\Phi_1(x, x, \hat{\Delta})$ of the heat-kernel expansion of $\hat{\Delta}$ (cf. [BGV]). Thus the problem of computing $\mathrm{Res}(\hat{\Delta}^{-n+1})$ is transformed into the problem of computing F.

For an arbitrary generalized laplacian, this might prove to be difficult. However, in the case where $E = S$ and the generalized laplacian $\hat{\Delta}$ is the square $\widetilde{D}^2$ of a Dirac operator associated to an arbitrary metric connection $\widetilde{\nabla}$ on TM, a constructive version of (1.2) can be proved. This will be shown in Section 2 . Because of its close relationship to the well-known Lichnerowicz formula (cf. [L]) we call our decomposition formula a 'generalized Lichnerowicz formula'. We understand it as being intrinsic to the Dirac operators studied in this paper.

As already mentioned, we will compute the lagrangian of (1.1) in Section 3, using our generalized Lichnerowicz formula as the main technical tool. From a physical point of view, this lagrangian can be interpreted as defining a modified (euclidian) Einstein–Cartan theory.

2. A generalized Lichnerowicz formula

Let M be a spin manifold with $\dim M = 2n$ and let us denote its Riemannian metric by g. The Levi-Civita connection $\nabla : \Gamma(TM) \to \Gamma(T^*M \otimes TM)$ on M induces a connection $\nabla^S : \Gamma(S) \to \Gamma(T^*M \otimes S)$ on the spinor bundle S which is compatible with the hermitian metric $\langle \cdot, \cdot \rangle_S$ on S. By adding an additional torsion term $t \in \Omega^1(M, \operatorname{End} TM)$ we obtain a new covariant derivative

$$\widetilde{\nabla} := \nabla + t \tag{2.1}$$

on the tangent bundle TM. Since t is really a one-form on M with values in the bundle of skew endomorphism $Sk(TM)$ (cf. [GHV]), $\widetilde{\nabla}$ is in fact compatible with the Riemannian metric g and therefore also induces a connection $\widetilde{\nabla}^S = \nabla^S + T$ on the spinor bundle. Here $T \in \Omega^1(M, \operatorname{End} S)$ denotes the 'lifted' torsion term $t \in \Omega^1(M, \operatorname{End} TM)$. However, in general this induced connection $\widetilde{\nabla}^S$ is neither compatible with the hermitian metric $\langle \cdot, \cdot \rangle_S$ nor compatible with the Clifford action on S. With respect to a local orthonormal frame $\{e_a\}_{1 \le a \le 2n}$ of $TM|_{U \subset M}$ we have

$$\nabla_c\, e_b = \omega^a{}_{bc} e_a, \qquad t := t^a{}_{bc}\; e_a \otimes e^b \otimes e^c,$$
$$\nabla^S s_l = \tfrac{1}{4}\, \gamma^a \gamma^b s_l \otimes \omega_{abc} e^c, \qquad \widetilde{\nabla}^S s_l = \tfrac{1}{4}\, \gamma^a \gamma^b s_l \otimes \left(\omega_{abc} + t_{abc}\right) e^c,$$

where $\omega^a{}_{bc}$ denotes the components of the Levi-Civita connection, $\{e^a\}_{1 \le a \le 2n}$ the corresponding dual frame of $\{e_a\}_{1 \le a \le 2n}$ and $\{s_l\}_{1 \le l \le \dim S}$ a local frame of $S|_U$. Note that we use the following conventions:

$$\{\gamma^a, \gamma^b\} = -2\eta^{ab}, \qquad [\gamma^a, \gamma^b] = 2\gamma^{ab}$$

for the representation $\gamma : C_{\mathbb{C}}(T^*M) \to \operatorname{End} S$ of the complexified Clifford algebra of T^*M on the spinor bundle.

We now define by $\widetilde{D} := \gamma^\mu \widetilde{\nabla}^S_\mu$ a first-order operator $\widetilde{D} : \Gamma(S) \to \Gamma(S)$ associated to the metric connection $\widetilde{\nabla}$. Since $\widetilde{D}$ satisfies the relationship $[\widetilde{D}, f] = \gamma^\mu (\partial f / \partial x^\mu)$ for all $f \in C^\infty(M)$ this operator $\widetilde{D}$ is a Dirac operator, i.e. its square $\widetilde{D}^2$ is a generalized laplacian (cf. [BGV]). Note that $\widetilde{D}$ is also the most general Dirac operator on the spinor bundle S corresponding to a metric connection $\widetilde{\nabla}$ on TM.

According to the well-known Ricci lemma (cf. [GHV]) there is a one-to-one correspondence between metric connections on TM and the elements of $\Omega^1(M, Sk(TM))$. Consequently, the set of all such Dirac operators acting on sections of the spinor bundle S over M is parametrized by $t \in \Omega^1(M, Sk(TM))$.

For the square of the Dirac operator $\widetilde{D}$ we get the following standard decomposition:

$$\widetilde{D}^2 = -g^{\mu\nu} \widetilde{\nabla}^S_\mu \widetilde{\nabla}^S_\nu + \gamma^\mu [\widetilde{\nabla}^S_\mu, \gamma^\nu] \widetilde{\nabla}^S_\nu + \tfrac{1}{2}\, \gamma^\mu \gamma^\nu [\widetilde{\nabla}^S_\mu, \widetilde{\nabla}^S_\nu]. \tag{2.2}$$

If $t = 0$, which means that $\widetilde{\nabla}$ is identical with the Levi-Civita connection ∇, Eq. (2.2) is the first step to compute the well-known Lichnerowicz formula of D^2, cf. [L]. Note that none of the first two terms of (2.2) is covariant in itself but only their sum. Using (2.1) we can, however, rearrange the decomposition (2.2) such that each term is manifestly covariant.

Moreover, the derivations in the decomposition will then be arranged according to their degree.

Lemma 2.1. *Let M be a spin manifold, ∇ the Levi-Civita connection on TM and $\widetilde{\nabla}$ defined by $\widetilde{\nabla} := \nabla + t$ with $t \in \Omega^1(M, Sk(TM))$. Then the square $\widetilde{D}^2$ of the Dirac operator $\widetilde{D}$ on the spinor bundle S associated to $\widetilde{\nabla}$ decomposes as*

$$\widetilde{D}^2 = \Delta^{\nabla} - B^{\mu}\nabla^S_{\mu} + F', \tag{2.3}$$

where $B \in \Gamma(TM \otimes \operatorname{End} S)$ and $F' \in \Gamma(\operatorname{End} S)$ are defined by

$$B^a := 2T^a - \gamma^c[T_c, \gamma^a], \tag{2.4}$$

$$F' := \tfrac{1}{4}R^{\nabla} \cdot 1_{\operatorname{End} S} + \gamma^a\gamma^b({}'\nabla_a T_b) + \gamma^a T_a \gamma^b T_b. \tag{2.5}$$

Furthermore R^{∇} denotes the scalar curvature and $\Delta^{\nabla} := \eta^{ab}(\nabla^S_a\nabla^S_b - \nabla^S_{\nabla_a e_b})$ the horizontal laplacian on the spinor bundle corresponding to the Levi-Civita connection ∇ with respect to a local orthonormal frame $\{e_a\}_{1\le a\le 2n}$.

Proof. By inserting $\widetilde{\nabla}^S_{\mu} = \nabla^S_{\mu} + T_{\mu}$ in (2.2) and using the compatibility of the connection ∇^S with the Clifford action, so that $[\nabla^S_{\mu}, \gamma^{\sigma}] = -\gamma^{\nu}\Gamma^{\sigma}_{\nu\mu}$ we get

$$\begin{aligned}\widetilde{D}^2 = {}&\Delta^{\nabla} + \tfrac{1}{2}\gamma^{\mu\nu}[\nabla^S_{\mu}, \nabla^S_{\nu}] - g^{\mu\nu}([\nabla^S_{\mu}, T_{\nu}] - \Gamma^{\sigma}_{\nu\mu}T_{\sigma})\\ &-g^{\mu\nu}T_{\mu}T_{\nu} - g^{\mu\nu}(2T_{\mu}\nabla^S_{\nu}) + \gamma^{\mu}[T_{\mu}, \gamma^{\nu}]\nabla^S_{\nu} + \tfrac{1}{2}\gamma^{\mu\nu}[T_{\mu}, T_{\nu}]\\ &+\gamma^{\mu\nu}[\nabla^S_{\mu}, T_{\nu}] + \gamma^{\mu}[T_{\mu}, \gamma^{\nu}]T_{\nu}.\end{aligned} \tag{2.6}$$

With the help of the Clifford relation $\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = -2g^{\mu\nu}$ and the first Bianchi identity $R_{jkli} + R_{klij} + R_{lijk} = 0$ one can identify the second term in (2.6) with the 'usual' Lichnerowicz term:

$$\tfrac{1}{2}\gamma^{\mu\nu}[\nabla^S_{\mu}, \nabla^S_{\nu}] = \tfrac{1}{4}R^{\nabla} \cdot 1_S.$$

If we write $\gamma^{\mu}[T_{\mu}, \gamma^{\nu}]\nabla_{\nu} = g^{\mu\nu}\gamma^{\sigma}[T_{\sigma}, \gamma_{\mu}]\nabla_{\nu}$, we see that B^{ν} is given by the sum of the fifth together with the sixth term on the right-hand side. Furthermore we have the identities

$$\begin{aligned}&-g^{\mu\nu}([\nabla^S_{\mu}, T_{\nu}] - \Gamma^{\sigma}_{\mu\nu}T_{\sigma}) = -g^{\mu\nu}((\nabla^{\operatorname{End} S}_{\mu}T_{\nu}) - \Gamma^{\sigma}_{\mu\nu}T_{\sigma}) = -g^{\mu\nu}({}'\nabla_{\mu}T_{\nu})\\ &\gamma^{\mu\nu}[\nabla^S_{\mu}, T_{\nu}] = \gamma^{\mu\nu}((\nabla^{\operatorname{End} S}_{\mu}T_{\nu}) - \Gamma^{\sigma}_{\mu\nu}T_{\sigma}) = \gamma^{\mu\nu}({}'\nabla_{\mu}T_{\nu}).\end{aligned} \tag{2.7}$$

Here ${}'\nabla : \Gamma(\operatorname{End} S \otimes T^*M) \to \Gamma(T^*M \otimes \operatorname{End} S \otimes T^*M)$ denotes the induced connection ${}'\nabla := \nabla^{\operatorname{End} S} \otimes 1_{T^*M} + \mathbb{I}_{\operatorname{End} S} \otimes \nabla$ on the tensor bundle $\operatorname{End} S \otimes T^*M$. Because ∇ respects the Clifford relation this means that

$${}'\nabla_{\mu}T_{\nu} = \tfrac{1}{4}\gamma^{ab}\nabla^{T^*M\otimes T^*M\otimes T^*M}_{\mu} t_{ab\nu} \equiv \tfrac{1}{4}\gamma^{ab}t_{ab\nu;\mu}.$$

Due to the fact that $\gamma^{\mu\nu} - g^{\mu\nu} = \gamma^{\mu}\gamma^{\nu}$ we obtain our result. □

It is well known (see [BGV]) that given any generalized laplacian $\hat{\Delta}$ on a hermitian bundle E over M, there exists a connection $\hat{\nabla}^E$ on E and a section F of the endomorphism bundle $\mathrm{End}(E)$, such that $\hat{\Delta}$ decomposes as

$$\hat{\Delta} = \Delta^{\hat{\nabla}^E} + F. \tag{2.8}$$

As we have mentioned before, this statement does not offer any possibility of calculating the endomorphism F explicitly. Since it can be shown (cf. [BGV]), however, that

$$\Phi_1(x, x, \hat{\Delta}) = \tfrac{1}{6} R^{\nabla} \cdot 1_E - F,$$

it is evident that F plays a leading role in the computation of the subleading term $\Phi_1(x, x, \hat{\Delta})$ in the asymptotic expansion of the heat kernel of $\hat{\Delta}$. Moreover, by the main theorem of [KW][2]

$$\mathrm{Res}\big(\hat{\Delta}^{-n+1}\big) = \frac{2n-2}{2} \int_M * \operatorname{tr}\big(\Phi_1(x, x, \hat{\Delta})\big), \tag{2.9}$$

this endomorphism F also determines the Wodzicki residue of $\hat{\Delta}^{-n+1}$ which defines gravity actions in the case of $\hat{\Delta} = \widetilde{D}^2$.

We shall now prove a theorem which enables us to compute F explicitly in the case of $E = S$ and $\hat{\Delta} := \widetilde{D}^2$.

Theorem 2.2. *Let the hypotheses be the same as in Lemma* 2.1 *and let* $\hat{\Delta} = \widetilde{D}^2$ *be the square of the Dirac operator* $\widetilde{D}$ *associated to* $\widetilde{\nabla}$. *Then the covariant derivative* $\hat{\nabla}^S$ *and the endomorphism* $F \in \Gamma(\mathrm{End}\, S)$ *in the decomposition* (2.8) *are defined as follows*:

$$\hat{\nabla}^S := \nabla^S + \hat{T}, \tag{2.10}$$

$$F := F' + \aleph. \tag{2.11}$$

With respect to a local orthonormal frame $\{e_a\}_{1 \le a \le n}$ *of* TM, $\hat{T} \in \Omega^1(M, \mathrm{End}\, S)$ *and* $\aleph \in \Gamma(\mathrm{End}\, S)$ *are explicitly given by*

$$\hat{T}_a = T_a - \tfrac{1}{2} \gamma^b [T_b, \gamma_a] \tag{2.12}$$

$$\aleph = {}'\nabla_a \hat{T}^a + \hat{T}_a \hat{T}^a, \tag{2.13}$$

where $F' \in \Gamma(\mathrm{End}\, S)$ *is the endomorphism* (2.5) *of Lemma* 2.1.

Proof. The main ingredient of this proof is the global decomposition formula (2.3) of $\widetilde{D}^2$ as given in Lemma 2.1. Concerning the case of $\hat{\Delta} = \widetilde{D}^2$, Eq. (2.3) is but an alternative version of (2.8). We can therefore prove the theorem by inserting (2.10)–(2.13) into Eq. (2.8) which then is identical with (2.3). □

[2] We denote by $*$ the Hodge-star operator associated to the Riemannian metric g.

Thus we obtain the following formula for the square of the Dirac Operator $\widetilde{D}$:

$$\widetilde{D}^2 = \Delta^{\hat{\nabla}^S} + \tfrac{1}{4}\, R \cdot \mathbf{1}_S + \gamma^{ab}({}'\nabla_a T_b) + \tfrac{1}{2}\, \gamma^{ab}[T_a, T_b] + \tfrac{1}{2}\, [\gamma^a[T_a, \gamma^b], T_b]$$
$$-\tfrac{1}{2}\, \gamma^b[({}'\nabla_a T_b), \gamma^a] + \tfrac{1}{4}\, \eta_{ab}\gamma^c[T_c, \gamma^a]\gamma^d[T_d, \gamma^b]. \tag{2.14}$$

In the case of $t = 0$, i.e. $T = 0$, this decomposition obviously reduces to the usual Lichnerowicz formula $D^2 = \Delta^{\nabla} + \frac{1}{4}R^{\nabla} \cdot \mathbb{I}_S$. Consequently, we call (2.14) a 'generalized Lichnerowicz formula'.

Notice that one has to take into account that in general it is impossible to find any $Sk(TM)$-valued one-form $\hat{t} \in \Omega^1(M, Sk(TM))$ such that the endomorphism part $\hat{T}_X \in \operatorname{End} S$ of $\hat{T}$ corresponds to $\hat{t}_X \in Sk(TM)$ for all $X \in \Gamma(TM)$. Hence, $\hat{\nabla}^S$ is generally not induced by any metric connection $\hat{\nabla}$ on TM. However, if $t \in \Omega^1(M, Sk(TM))$ is totally antisymmetric we obtain the following.

Lemma 2.3. *Let M be a spin manifold, ∇ the Levi-Civita connection on TM and $\widetilde{\nabla}$ defined by $\widetilde{\nabla} := \nabla + t$ where $t \in \Omega^1(M, Sk(TM))$ is totally antisymmetric. Then $\hat{T} = 3T$ and consequently $\hat{\nabla}^S = \nabla^S + 3T$.*

This can simply be derived from the definiton (2.12) of $\hat{T}$.

3. Euclidian gravity

According to Connes [C] there exists a link between the usual Dirac operator $D := \gamma^{\mu}\nabla^S_{\mu}$ on the spinor bundle S of a four-dimensional spin manifold M associated to the Levi-Civita connection and the euclidian Einstein–Hilbert gravity action via the Wodzicki residue $\operatorname{Res}(D^{-2})$ of the inverse of D^2. This has been explicitly verified in [K2]. Moreover, as already mentioned, the main theorem of [KW] states that the Wodzicki residue $\operatorname{Res}(\hat{\Delta}^{-n+1})$ of any generalized laplacian $\hat{\Delta}$ acting on sections of an hermitian vector bundle E over an even-dimensional manifold M with $\dim M = 2n \geq 4$ can be identified with

$$\frac{2n-2}{2} \int_M * \operatorname{tr}\big(\Phi_1(x, x, \hat{\Delta})\big).$$

Again $\Phi_1(x, x, \hat{\Delta})$ denotes the subleading term of the asymptotic expansion of the heat kernel of $\hat{\Delta}$. In this sense a gravity action can be defined by an arbitrary Dirac operator $\widetilde{D} := \gamma^{\mu}\widetilde{\nabla}^S_{\mu}$ on S associated to a metric connection $\widetilde{\nabla}$ on the base M, this means

$$I_{\mathrm{GR}}(\widetilde{D}) := -\frac{1}{2^n} \int_M * \operatorname{tr}\big(\Phi_1(x, x, \widetilde{D}^2)\big). \tag{3.1}$$

Here $2^n = \dim_{\mathbb{C}} S$ is the complex dimension of the spinor bundle. By using our generalized Lichnerowicz formula (2.14), we can now easily compute $\operatorname{tr}\big(\Phi_1(x, x, \widetilde{D}^2)\big)$. All that remains to be done is to take traces of γ-matrices. Thus, we obtain:

Lemma 3.1. *Let M be a spin manifold with* $\dim M = 2n$ *even and* $\widetilde{D} : \Gamma(S) \to \Gamma(S)$ *the Dirac operator on the spinor bundle S associated to a metric connection* $\widetilde{\nabla} := \nabla + t$ *as above. Then*

$$-\frac{1}{2^n}\,\mathrm{tr}\big(\Phi_1(x,x,\widetilde{D}^2)\big) = \frac{1}{12}\,R^{\nabla} + \frac{1}{2^n}\,\big(-t_{abc}t^{abc} + 2t_{abc}t^{acb}\big) \tag{3.2}$$

with respect to a local orthonormal frame of TM.

Note that this result (3.2) holds independently of whether or not the corresponding Dirac operator $\widetilde{D}$ is self-adjoint with respect to the hermitian metric on the spinor bundle S. In the special case of the torsion tensor being totally anti-symmetric, (3.2) reduces to

$$-\frac{1}{2^n}\,\mathrm{tr}\big(\Phi_1(x,x,\widetilde{D}^2)\big) = \frac{1}{12}\,R^{\nabla} - \frac{3}{2^n}\,t_{[abc]}t^{[abc]}$$

as already shown in [KW].

In order to find out whether (3.2) defines a pure (euclidian) Einstein–Cartan theory [3] we express the right-hand side of (3.2) by the scalar curvature $R^{\widetilde{\nabla}}$ of $\widetilde{\nabla}$. Using the well-known formula $\mathbf{R}^{\widetilde{\nabla}} = \mathbf{R}^{\nabla} + \mathrm{d}^{\nabla}t + \frac{1}{2}\,[t \wedge t]$, where $\mathbf{R}^{\widetilde{\nabla}} \in \Omega^2(M, \mathrm{End}\,TM)$ denotes the curvature of $\widetilde{\nabla}$ and d^{∇} is the exterior covariant derivative corresponding to the Levi-Civita connection ∇, we can rewrite (3.2) as follows

$$\begin{aligned}-\frac{1}{2^n}\,\mathrm{tr}\big(\Phi_1(x,x,\widetilde{D}^2)\big) = {} & \frac{1}{12}\,R^{\widetilde{\nabla}} + \frac{1}{12}\,t_{abc}t^{bca} - \frac{1}{2^n}\,t_{abc}t^{abc} + \frac{1}{2^{n-1}}\,t_{abc}t^{acb} \\ & + \frac{1}{12}\,t_{ab}{}^{b}t^{a}{}_{c}{}^{c} - \frac{1}{12}\,\nabla_{\mu}t^{\mu}{}_{a}{}^{a}.\end{aligned} \tag{3.3}$$

Without additional mater fields, our result (3.2) obviously reduces to the usual Einstein theory of gravity. Hence, we obtain a result similar to that in [KW]. We also conclude from (3.3) that it is not possible to obtain a 'pure' Einstein–Cartan theory from the square of an arbitrary Dirac operator $\widetilde{D}$ associated to a metric connection on M by using the Wodzicki residue.

4. Conclusion

In this paper we have proved a generalized version of the well-known Lichnerowicz formula for the most general Dirac operator $\widetilde{D}$ on the spinor bundle of an even-dimensional spin manifold M associated to a metric connection $\widetilde{\nabla}$ on TM. Applying this formula, the subleading term $\Phi_1(x,x,\widetilde{D}^2)$ of the heat-kernel expansion of $\widetilde{D}^2$ is easy to compute. According to [KW], the trace of this term plays a key role in the definition of a (euclidian) gravity action $I_{\mathrm{GR}}(\widetilde{D})$ in the context of the non-commutative differential geometry introduced by Connes [C]. This gravity action can be interpreted as defining a modified Einstein–Cartan theory.

[3] By an Einstein–Cartan theory we mean a gravity theory based on the Einstein–Hilbert action.

Finally, we would like to add that it is also possible to derive a combined Einstein–Hilbert/Yang–Mills lagrangian from an appropriately defined Dirac operator by using similar techniques. Moreover, this Dirac operator can be considered as a deformation of the well-known Dirac–Yukawa operator. This will be shown in a forthcoming paper [AT].

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References

[AT] T. Ackermann and J. Tolksdorf, The generalized Lichnerowicz formula and analysis of Dirac operators, Preprint CPT-95/P.3166 and Mannheimer Manuskripte 186 (1995), to appear in J-Reine Angew. Math.

[BGV] N. Berline, E. Getzler and M. Vergne, *Heat Kernels and Dirac Operators* (Springer, Berlin, 1992).

[C] A. Connes, Non-commutative geometry and physics, IHES preprint (1993).

[CL] A. Connes and J. Lott, Particle models and non-commutaive geometry, Nucl. Phys. B Proc. Supp. 18B (1990) 29–47.

[GHV] W. Greub, S. Halperin and R. Vanstone, *Connections, Curvature and Cohomology*, Vol. 1 (Academic press, New York, 1976).

[KW] W. Kalau and M. Walze, Gravity, non-commutative geometry and the Wodzicki residue, J. Geom. Phys. 16 (1995) 327–344.

[L] A. Lichnerowicz, Spineurs harmonique, C. R. Acad. Sci. Paris Sér. A 257 (1963) 7–9.

[K1] D. Kastler, A detailed account of Alain Connes' version of the standard model in non-commutative differential geometry I, II, Rev. Math. Phys. 5 (1993) 477–532.

[K2] D. Kastler, The Dirac operator and gravitation, Comm. Math. Phys. 166 (1995) 633–643.